

ZERO-DENSITY ESTIMATE FOR MODULAR FORM L -FUNCTIONS IN WEIGHT ASPECT

BOB HOUGH

ABSTRACT. Considering the family of L -functions $\{L(s, f)\}_{f \in H_k}$ where H_k is the set of weight k Hecke-eigen cusp forms for $SL_2(\mathbb{Z})$, we prove a zero density estimate near the central point, valid as the weight $k \rightarrow \infty$.

1. INTRODUCTION

In the analytic theory of L -functions, it is sometimes possible to circumvent assumption of the Riemann Hypothesis by applying certain zero density arguments. Briefly, one argues that in a family of L -functions that is sufficiently “spectrally complete”, the number of zeros of functions in the family to the right of the half-line is comparatively few. Historically, zero density questions were first considered with respect to the Riemann zeta function $\zeta(\frac{1}{2} + \sigma + it)$ as the parameter t varied, and the first result along these lines could be said to be the Hadamard-de la Vallée-Poussin zero-free region. Later investigations focused on the number

$$N(\sigma, T) = \#\{\rho = \frac{1}{2} + \beta + i\gamma : \zeta(\rho) = 0, \sigma < \beta, 0 < \gamma < T\},$$

proving that this number decayed in the power of T with increasing $\sigma > 0$. The strongest result in this direction is due to Ingham [8]

$$N(\sigma, T) = O(T^{3(\frac{1}{2}-\sigma)/(\frac{3}{2}-\sigma)} \log^5 T).$$

Selberg [19] made a major contribution to this theory, proving the uniform bound

$$N(\sigma, T) \ll T^{1-\frac{\sigma}{4}} \log T,$$

in $0 \leq \sigma \leq \frac{1}{2}$. The crucial feature of this estimate is that the power of $\log T$ matches the true order in the number of zeros of ζ up to height T , so that the estimate is still useful even when σ is on the order of $\frac{1}{\log T}$. This formed one of the key analytic ingredients in Selberg’s unconditional proof that the real and imaginary parts of $\log \zeta(\frac{1}{2} + it)$ become normally distributed in large intervals $t \in [T, 2T]$.

Subsequent to his work on ζ , Selberg [18] proved an analogous zero density estimate in the family of Dirichlet L -functions to a large modulus q , with q rather than t thought of as the varying parameter. Using this estimate, he showed that for fixed t the argument of $L(\frac{1}{2} + it, \chi)$ becomes normally distributed as χ varies modulo q , for $q \rightarrow \infty$. More recently Luo [15] has given an analogue of Selberg’s bound in t -aspect, replacing ζ with the L -function of a fixed Hecke-eigen cusp form for $SL_2(\mathbb{Z})$:

$$N_f(\sigma, T) := \#\{\rho = \frac{1}{2} + \beta + i\gamma : L(\rho; f) = 0, \sigma < \beta, 0 < \gamma < T\} \ll_f T^{1-\frac{\sigma}{72}} \log T.$$

Together with earlier work of Bombieri and Hejhal [1], this established the asymptotic normality of $\log L(\frac{1}{2} + it; f)$ for $t \in [T, 2T]$, f fixed with $T \rightarrow \infty$. The purpose of this article is to prove a parallel extension of Selberg's Dirichlet L -function estimate but now for the family of L -functions associated to modular forms of large weight k . As in Selberg's work, an important aspect of our estimate is that it is uniform in k and for T in the ranges $\frac{1}{\log k} < T < k^\delta$, for some small $\delta > 0$. This plays a crucial role in the author's related paper [7], where it is established, unconditionally, that varying f among Hecke-eigenforms of weight k , $\log L(\frac{1}{2}; f)$ is bounded above by a quantity that is asymptotically normal as $k \rightarrow \infty$. One further piece of context: Kowalski and Michel [13] have proven another extension of Selberg's theorem to the family of weight 2 modular forms of large prime level q , and Conrey and Soundarajan [2] (real Dirichlet L -functions) and Ricotta [16] (Rankin-Selberg L -functions) have given related estimates, each with applications to non-vanishing. Suitably modified, our estimate has similar applications, but we do not pursue them here.

To state our density result more precisely, let S_k denote the space of weight k cusp forms for the full modular group $\Gamma = SL(2, \mathbb{Z})$ and let H_k be the basis of forms in S_k that are simultaneous eigenfunctions of all the Hecke operators. Write the Fourier expansion of $f \in H_k$ as

$$f = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} \lambda_f(n) e(nz).$$

We normalize $f \in H_k$ so that $\lambda_f(1) = 1$ ¹. The L -function associated to $f \in H_k$ is

$$(1) \quad L(s; f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1}, \quad \Re(s) > 1.$$

This is a degree two L -function with completed L -function

$$\Lambda(s; f) = (2\pi)^{-s} \Gamma(s + \frac{k-1}{2}) L(s; f)$$

satisfying the self-dual functional equation

$$\Lambda(s; f) = i^k \Lambda(1-s; f).$$

In particular, with our normalization the Riemann Hypothesis asserts that all zeros ρ of $\Lambda(s; f)$ satisfy $\Re(\rho) = \frac{1}{2}$.

For $f \in H_k$ and T growing, but small compared to \sqrt{k} , the number of zeros ρ of $L(s; f)$ with $0 < \Im(\rho) < T$ is $\sim \frac{T}{2\pi} \log k$. Thus the density of zeros of $L(s; f)$ near the central point $s = \frac{1}{2}$ is $\frac{\log k}{2\pi}$. Our main result says that on average over the family \mathcal{F}_k there are very few forms with zeros with small imaginary part and real part to the right of $\frac{1}{2} + \frac{C}{\log k}$.

Theorem 1.1. *Let $\frac{2}{\log k} < \sigma < \frac{1}{2}$. For some sufficiently small $\delta, \theta > 0$ we have uniformly in $\frac{10}{\log k} < T < k^\delta$*

$$N(\sigma, T) \stackrel{\text{def}}{=} \frac{1}{|H_k|} \sum_{f \in H_k} \# \left\{ L\left(\frac{1}{2} + \beta + i\gamma\right) = 0 : \sigma < \beta, |\gamma| < T \right\} = O(T k^{-\theta\sigma} \log k).$$

¹In particular, in our normalization Deligne's bound [3] reads $|\lambda_f(n)| \leq d(n)$, the number of divisors of n .

The main new analytic ingredient of our theorem is the following asymptotic evaluation of the harmonic twisted second moment of $L(s; f)$, which may be of independent interest.

Theorem 1.2. *Let $\sigma > 0$, $0 \neq |t| < k^{\frac{1}{4}}$ and $\ell < k^{\frac{1}{3}}$ be squarefree. We have the following formula for the harmonic twisted second moment.*

$$\begin{aligned} & \sum_{f \in H_k}^h \lambda_f(\ell) |L(\frac{1}{2} + \sigma + it; f)|^2 \\ &= \zeta(1 + 2\sigma) \frac{\tau_{it}(\ell)}{\ell^{\frac{1}{2} + \sigma}} + \zeta(1 - 2\sigma) \left(\frac{k}{4\pi}\right)^{-4\sigma} \frac{\tau_{it}(\ell)}{\ell^{\frac{1}{2} - \sigma}} \\ &+ i^k \zeta(1 + 2it) \left(\frac{k}{4\pi}\right)^{-2\sigma + 2it} \frac{\tau_{\sigma}(\ell)}{\ell^{\frac{1}{2} + it}} + i^k \zeta(1 - 2it) \left(\frac{k}{4\pi}\right)^{-2\sigma - 2it} \frac{\tau_{\sigma}(\ell)}{\ell^{\frac{1}{2} - it}} \\ &+ O\left(\ell^{1+\sigma} k^{-\frac{1}{2} - 2\sigma + \epsilon}\right). \end{aligned}$$

The harmonic average (indicated by \sum^h) means that forms $f \in H_k$ are counted with the weight

$$w_f = \frac{(4\pi)^{1-k} \Gamma(k-1)}{\langle f, f \rangle},$$

which appears in the Petersson trace formula. Harmonic averages similar to this one have an extensive history; see for instance [14], [5], [6] and references therein. Our proof is most noteworthy for the fact that the evaluation of main terms goes “beyond the diagonal” and yet is not too difficult. After applying the Petersson trace formula and Voronoi summation to the resulting sums of Kloosterman sums, the off-diagonal main term arises as the Fourier transform of the relevant function at zero, and the remaining integrals against Bessel functions are error terms. The analysis of these error terms involves integrating against the Bessel function $J_{k-1}(x)$ near its transition region, and this is bounded in a similar way to an analysis of the twisted first moment of $L(\frac{1}{2}, \text{sym}^2 f)$ in [12].

2. OUTLINE OF PROOF

The method of proof of Theorem 1.1 is the same as in Selberg’s original work on Dirichlet L -functions; in particular, we appeal to the following version of the argument principle introduced there.

Lemma 2.1. *Let ω be a holomorphic function, non-zero in the half plane $\Re(s) > W$. Let \mathcal{B} be the rectangular box $|\Im(s)| \leq H$, $W_0 \leq \Re(s) \leq W_1$ with $W_0 < W < W_1$. Then*

$$\begin{aligned} & 4H \sum_{\substack{\beta + i\gamma \in \mathcal{B} \\ \omega(\beta + i\gamma) = 0}} \cos\left(\frac{\pi\gamma}{2H}\right) \sinh\left(\frac{\pi(\beta - W_0)}{2H}\right) \\ &= \int_{-H}^H \cos\left(\frac{\pi t}{2H}\right) \log |\omega(W_0 + it)| dt - \Re \int_{-H}^H \cos\left(\pi \frac{W_1 - W_0 + it}{2iH}\right) \log \omega(W_1 + it) dt \\ &+ \int_{W_0}^{W_1} \sinh\left(\frac{\pi(\alpha - W_0)}{2H}\right) \log |\omega(\alpha + iH)\omega(\alpha - iH)| d\alpha. \end{aligned}$$

The fundamental proposition that we prove is the following.

Proposition 2.2. *There exist holomorphic mollifying functions $\{M(s; f)\}_{f \in H_k}$ satisfying $M(\bar{s}) = \overline{M(s)}$, such that for sufficiently small positive δ and θ , uniformly in $|t| < k^\delta$, $\frac{1}{\log k} \leq \sigma \leq 1$*

$$\frac{1}{|H_k|} \sum_{f \in H_k} \left| M\left(\frac{1}{2} + \sigma + it; f\right) L\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 \leq 1 + O(k^{-\theta\sigma})$$

and for all t ,

$$M\left(\frac{3}{2} + it; f\right) L\left(\frac{3}{2} + it; f\right) = 1 + O(k^{-\theta}).$$

To deduce Theorem 1.1 from this Proposition, apply the lemma to the holomorphic functions $M(s; f)L(s; f)$ with box bounded by $\frac{1}{2} + \frac{1}{\log k} \pm 2iT$ and $\frac{3}{2} \pm 2iT$. The special feature of the lemma which permits uniformity even for small $T \asymp \frac{1}{\log k}$, is that only the real part of the logarithm appears in the part of the integral contained in the critical strip, so that this part may be bounded using the second moment estimate of the Proposition.

The further details of the deduction of Theorem 1.1 are not difficult, and may be found both in Selberg's original argument, and in the treatments in [2] and [13]. In the remainder of the paper we are concerned with the proof of the Proposition, which takes place in three stages: first we calculate the harmonic twisted moment, proving Theorem 1.2. Next we mollify the second moment with respect to the harmonic weights. Finally we remove the harmonic weights via the method of [13].

3. SOME LEMMAS

Lemma 3.1 (Hecke Relations). *For each Hecke eigenform f , the Fourier coefficients of f satisfy the relation*

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

This is equivalent to the Euler product (1).

The basic orthogonality relation on H_k is the Petersson Trace Formula.

Lemma 3.2 (Petersson Trace Formula). *We have*

$$\sum_{f \in H_k}^h \lambda_f(m)\lambda_f(n) = \delta_{m=n} + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c} J_{k-1}\left(\frac{4\pi}{c} \sqrt{mn}\right)$$

Proof. See e.g. [9] p 360. □

Denote by

$$(2) \quad \tau_\nu(n) = \sum_{n_1 n_2 = n} \left(\frac{n_1}{n_2} \right)^\nu$$

the generalized divisor function. We will use the following version of the Voronoi summation formula.

Lemma 3.3. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be smooth with compact support. Let $c \geq 1$ and $(a, c) = 1$ with $ad \equiv 1 \pmod{c}$. We have*

$$\begin{aligned} \sum_{m=1}^{\infty} \tau_{it}(m) e\left(\frac{am}{c}\right) g(m) &= c^{2it-1} \zeta(1-2it) \int_0^{\infty} g(x) x^{-it} dx + c^{-2it-1} \zeta(1+2it) \int_0^{\infty} g(x) x^{it} dx \\ &\quad + \frac{1}{c} \sum_{n=1}^{\infty} \tau_{it}(n) e\left(\frac{-dn}{c}\right) \int_0^{\infty} g(x) J_{2it}^+\left(\frac{4\pi}{c} \sqrt{nx}\right) dx \\ &\quad + \frac{1}{c} \sum_{n=1}^{\infty} \tau_{it}(n) e\left(\frac{dn}{c}\right) \int_0^{\infty} g(x) K_{2it}^+\left(\frac{4\pi}{c} \sqrt{nx}\right) dx \end{aligned}$$

where

$$J_{\nu}^+(x) = \frac{-\pi}{\sin \frac{\pi\nu}{2}} (J_{\nu}(x) - J_{-\nu}(x)), \quad K_{\nu}^+(x) = 4 \cos \frac{\pi\nu}{2} K_{\nu}(x).$$

Proof. This is a slight modification of [9] Theorem 4.10. □

In bounding oscillatory integrals we make use of the following simple estimate ([20], Lemma 4.5)

Lemma 3.4. *Let $F(x), G(x)$ be real-valued functions on $[a, b]$ satisfying $\frac{F'(x)}{G(x)}$ is monotonic and $F''(x) > r > 0$, $|G(x)| \leq M$. Then*

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq \frac{8M}{\sqrt{r}}.$$

3.1. Facts concerning Bessel functions. The Bessel function of the first kind, $J_{\nu}(x)$, has Taylor series about zero given by

$$(3) \quad J_{\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{\nu+2m}}{m! \Gamma(\nu+1+m)}.$$

Differentiating, one obtains the relation

$$(4) \quad J'_{\nu}(x) = \frac{1}{2} (J_{\nu+1}(x) - J_{\nu-1}(x)).$$

The Mellin Transform is given by

$$(5) \quad \int_0^{\infty} J_{k-1}(x) x^{s-1} dx = 2^{s-1} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)}.$$

For x small and ν fixed, $J_{\nu}(x)$ is well approximated by the first term of its Taylor expansion:

$$(6) \quad J_{\nu}(x) = \frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma(\nu)} + O_{\nu}(x^{\nu+1}).$$

Taking more terms of the expansion leads to the bound ([17], p 5)

$$(7) \quad |J_{k-1}(x)| \leq \frac{\left(\frac{x}{2}\right)^{k-1}}{\Gamma(k-1)} e^{\frac{x}{2}}, \quad x < 2k.$$

In particular, if $x < \frac{k}{10}$ then $J_{k-1}(x) < e^{-k}$. The Bessel function of the second kind, $Y_\nu(x)$, is defined by

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}.$$

For x small and $\nu > 0$ fixed we obtain

$$(8) \quad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu.$$

We also need the modified Bessel function of the third kind, $K_\nu(x)$. For small x and fixed positive ν , it is approximated by

$$(9) \quad K_\nu(x) \sim \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu.$$

When x is large, $x > |\nu|^2$ (ν possibly complex) asymptotic evaluations are given by ([4] p. 85)

$$(10) \quad J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \left[1 - \frac{P(\nu)}{128x^2}\right] - \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \frac{\nu^2 - \frac{1}{4}}{2x} + O\left(\frac{1 + |\nu|^6}{x^3}\right)$$

$$P(\nu) = 16\nu^4 - 40\nu^2 + 9$$

$$(11) \quad K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + O\left(\frac{1 + |\nu|^2}{x}\right)\right].$$

The asymptotic evaluation of Y_ν is the same as for J_ν , except that the places of \cos and \sin are interchanged. Also,

$$(12) \quad J_\nu^+(x) = -\sqrt{\frac{2\pi}{x}} \sin\left(x - \frac{\pi}{4}\right) \left[1 - \frac{P(\nu)}{128x^2}\right] - \pi \cos\left(x - \frac{\pi}{4}\right) \frac{\nu^2 - \frac{1}{4}}{2x} + O\left(\frac{1 + |\nu|^6}{x^3}\right).$$

In the transition region, where x is comparable in size compared to k , Watson's formulas ([4], p. 89) are, for $x < k$,

$$J_k(x) = \frac{we^{ka}}{\pi\sqrt{3}} K_{1/3}\left(\frac{kx^3}{3}\right) + O(k^{-1}); \quad a = k\left(w + \frac{w^3}{3} - \tanh^{-1} w\right), \quad w = \sqrt{1 - \frac{x^2}{k^2}}$$

and for $x > k$,

$$J_k(x) = \frac{w}{\sqrt{3}} \left[J_{1/3}\left(\frac{kx^3}{3}\right) \cos\left(\delta + \frac{\pi}{6}\right) - Y_{1/3}\left(\frac{kx^3}{3}\right) \sin\left(\delta + \frac{\pi}{6}\right) \right] + O(k^{-1});$$

$$\delta = k\left(w - \frac{w^3}{3} - \tan^{-1} w\right), \quad w = \sqrt{\frac{x^2}{k^2} - 1}.$$

Note that

$$x = k + k^\Delta \quad \Rightarrow \quad w \asymp k^{\frac{\Delta-1}{2}}.$$

For $|x - k| < k^{\frac{1}{3}}$, substituting the asymptotics (6), (8), (9) into Watson's formulas, we obtain the bound

$$(13) \quad J_k(x) \ll k^{-\frac{1}{3}}, \quad |x - k| < k^{-\frac{1}{3}}.$$

For $|x - k| > k^{\frac{1}{3}}$ we have $w \gg k^{-\frac{1}{3}}$. In this region, substituting the evaluations (10) and (11) we obtain

$$(14) \quad J_k(x) = \frac{e^{kw - k \tanh^{-1} w}}{\sqrt{2\pi kw}} [1 + O(k^{-1}w^{-3})] + O(k^{-1}), \quad x < k$$

$$(15) \quad J_k(x) = \sqrt{\frac{2}{\pi kw}} \cos\left(kw - k \tan^{-1} w - \frac{\pi}{4}\right) + O(k^{-1} + k^{-1}w^{-2}), \quad x > k.$$

One further consequence is the following simple lemma.

Lemma 3.5. *For any integer $k > 0$ and any $A < k^2$,*

$$\int_0^A |J_k(x)| dx \ll \sqrt{A}.$$

Proof. For $A < 1$ use the bound (7). In the range $A < k - k^{\frac{1}{2}}$ use (14) and $w \gg k^{-\frac{1}{4}}$ to deduce $J_k(x) \ll k^{-1} + e^{-\Omega(k^{\frac{1}{4}})} k^{-\frac{3}{8}}$ uniformly in $x < A$. For $k - k^{\frac{1}{2}} < x < k + k^{\frac{1}{2}}$ bound simply $J_k(x) = O(1)$. In the range $k + k^{\frac{1}{2}} < A < 2k$ use (15) to bound

$$\int_{k+k^{\frac{1}{2}}}^A |J_k(x)| \ll \frac{A}{k} + \int_{k+k^{\frac{1}{2}}}^A \frac{1}{\sqrt{kw}} + \frac{1}{kw^2} dx.$$

For $k < x < 2k$, $w \gg \sqrt{\frac{x-k}{k}}$, so the last integral is

$$\ll \int_{k^{\frac{1}{2}}}^{A-k} \frac{1}{(ky)^{\frac{1}{4}}} + \frac{1}{y} dy \ll \frac{A^{\frac{3}{4}}}{k^{\frac{1}{4}}} + \log k \ll \sqrt{A}.$$

Finally, for $x > 2k$, $w = \Omega(1)$ and so (15) says that $|J_k(x)| \ll \frac{1}{\sqrt{x}} + \frac{1}{k}$. □

With an eye toward applying Lemma 3.4 and with $x > k$ and $w = \sqrt{\frac{x^2}{k^2} - 1}$ as above, we record

$$(16) \quad \frac{\partial}{\partial x}(kw - k \tan^{-1} w) = \frac{kw}{x}, \quad \frac{\partial^2}{\partial x^2}(kw - k \tan^{-1} w) = \frac{k^2}{x^2(x^2 - k^2)^{\frac{1}{2}}}.$$

3.2. Approximate functional equation. Fix, once and for all, a smooth function $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying

- (1) $H(x) \equiv 1$ for $x \in [0, \frac{1}{2}]$
- (2) $H(x) + H(\frac{1}{x}) = 1$.

In particular, the Mellin transform $\hat{H}(s)$ has a single simple pole at 0 of residue 1, is odd, and satisfies the bounds

$$\hat{H}(s) \ll_A \frac{1}{s(s+1)\dots(s+A-1)}, \quad A = 1, 2, \dots$$

and, for $\Re(s) > 1$, $|\hat{H}(s)| \ll 2^{\Re(s)}$.

Recall Stirling's formula: for $\Re(z) > 1$

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{\ln 2\pi}{2} + \frac{1}{12z} + O(|z|^{-2})$$

and

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(|z|^{-2}).$$

It follows in a straightforward way that if $\Re(z)$ is large and $|\Delta| < \Re(z)^{\frac{1}{2}}$ then

$$(17) \quad \frac{\Gamma(z + \Delta)}{\Gamma(z)} = \exp \left(\Delta \log z + \frac{\Delta^2}{2z} + O(|\Delta||z|^{-1}) \right).$$

We now record an approximate formula for $|L(\frac{1}{2} + \sigma + it; f)|^2$.

Proposition 3.6 (Approximate functional equation). *We have*

$$(18) \quad |L(\frac{1}{2} + \sigma + it; f)|^2 = \sum_{d=1}^{\infty} \frac{1}{d^{1+2\sigma}} \sum_{m=1}^{\infty} \frac{\lambda_f(m) \tau_{it}(m)}{m^{\frac{1}{2} + \sigma}} \left(W_{k, \sigma + it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k, -\sigma + it}(md^2) \right).$$

Here

$$W_{k, \sigma + it}(\xi) = \frac{1}{2\pi i} \int_{(3)} \frac{1}{(4\pi^2 \xi)^s} \frac{\Gamma(\sigma + \frac{k}{2} + it + s) \Gamma(\sigma + \frac{k}{2} - it + s)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds$$

and

$$\tilde{W}_{k, -\sigma + it}(\xi) = \frac{1}{2\pi i} \int_{(3)} \frac{1}{(4\pi^2 \xi)^s} \frac{\Gamma(-\sigma + \frac{k}{2} + it + s) \Gamma(-\sigma + \frac{k}{2} - it + s)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds.$$

Proof. See [9] pp 97-100. □

The functions $W_{k, \sigma + it}$ and $\tilde{W}_{k, -\sigma + it}$ satisfy the following properties.

Lemma 3.7. *As functions of a real variable, both $W_{k,\sigma+it}$ and $\tilde{W}_{k,-\sigma+it}$ are real valued. For $t < k^{\frac{1}{4}}$ and $|\sigma| < 2$ we have*

$$W_{k,\sigma+it}(\xi) = 1 + O\left(\left(\frac{400\xi}{k^2}\right)^{k^{\frac{1}{4}}}\right), \quad W_{k,\sigma+it}(\xi) = O\left(\left(\frac{k^2}{80\xi}\right)^{k^{\frac{1}{4}}}\right),$$

$$\xi^j \left(\frac{\partial}{\partial \xi}\right)^j W_{k,\sigma+it}(\xi) \ll_j 1$$

and

$$\tilde{W}_{k,-\sigma+it}(\xi) = \frac{\Gamma(-\sigma + \frac{k}{2} + it)\Gamma(-\sigma + \frac{k}{2} - it)}{\Gamma(\sigma + \frac{k}{2} + it)\Gamma(\sigma + \frac{k}{2} - it)} + O\left(\left(\frac{400\xi}{k^2}\right)^{k^{\frac{1}{4}}}\right),$$

$$\tilde{W}_{k,-\sigma+it}(\xi) = O\left(\left(\frac{k^2}{80\xi}\right)^{k^{\frac{1}{4}}}\right), \quad \xi^j \left(\frac{\partial}{\partial \xi}\right)^j \tilde{W}_{k,-\sigma+it}(\xi) \ll_j k^{-4\sigma}.$$

Proof. Pair s and \bar{s} in the defining integrals to prove that W and \tilde{W} are real.

For the bounds on the functions, shift the contour to $\Re(s) = \pm k^{\frac{1}{4}}$ and estimate the ratio of Gamma factors using Stirling's approximation. The derivatives are bounded by estimating directly on the $\Re(s) = 0$ line. \square

4. TWISTED SECOND MOMENT

Proof of Theorem 1.2. From the approximate functional equation,

$$\sum_{f \in H_k}^h \lambda_f(\ell) |L(\frac{1}{2} + \sigma + it; f)|^2 = \sum_{m,d=1}^{\infty} \frac{\tau_{it}(m)}{m^{\frac{1}{2}+\sigma} d^{1+2\sigma}} \left[W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(md^2) \right] \sum_{f \in H_k}^h \lambda_f(\ell) \lambda_f(m)$$

Applying the Petersson inner product we obtain a diagonal term

$$\frac{\tau_{it}(\ell)}{\ell^{\frac{1}{2}+\sigma}} \sum_{d=1}^{\infty} \frac{1}{d^{1+2\sigma}} \left[W_{k,\sigma+it}(\ell d^2) + (4\pi^2 \ell d^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(\ell d^2) \right]$$

and an off-diagonal term

$$2\pi i^k \sum_{m,d=1}^{\infty} \frac{\tau_{it}(m)}{m^{\frac{1}{2}+\sigma} d^{1+2\sigma}} \left[W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(md^2) \right] \sum_{c=1}^{\infty} \frac{S(m, \ell; c)}{c} J_{k-1} \left(\frac{4\pi}{c} \sqrt{\ell m} \right).$$

Introducing the integrals defining W and \tilde{W} , the diagonal term is equal to

$$\frac{\tau_{it}(\ell)}{\ell^{\frac{1}{2}+\sigma}} \left\{ \frac{1}{2\pi i} \int_{(3)} \frac{\zeta(1+2\sigma+2s)}{(4\pi^2 \ell)^s} \frac{\Gamma(s+\sigma+\frac{k}{2}+it)\Gamma(s+\sigma+\frac{k}{2}-it)}{\Gamma(\sigma+\frac{k}{2}+it)\Gamma(\sigma+\frac{k}{2}-it)} \left(\hat{H}(s) + \hat{H}(s+2\sigma) \right) ds \right\}.$$

and this evaluates to the first two main terms, with an error that is $O(\frac{1}{k})$, by shifting the contour to the $\Re(s) = -\frac{1}{2}$ line.²

The two remaining main terms come from the off-diagonal sum, so we now work to isolate these terms. To begin, we exchange order of summation and discard those terms with $cd > 10000\sqrt{\ell}$, writing

$$2\pi i^k \sum_{cd < 10000\sqrt{\ell}} \frac{1}{cd^{1+2\sigma}} \sum_{m=1}^{\infty} \frac{\tau_{it}(m)S(m, \ell; c)}{m^{\frac{1}{2}+\sigma}} \left[W_{k, \sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k, -\sigma+it}(md^2) \right] \\ \times J_{k-1} \left(\frac{4\pi}{c} \sqrt{\ell m} \right).$$

This is justified because for $cd > 10000\sqrt{\ell}$ the sum over m is extremely small. Taking absolute values inside, those terms with $m \leq \frac{k^2 c}{\ell^{\frac{1}{2}} d}$ are bounded by applying the bound (7) for the Bessel function and bounding W and \tilde{W} by $O(1)$, yielding

$$\ll cd^{4\sigma} \frac{1}{\Gamma(k-1)} \sum_{m < k^2 c / d \ell^{\frac{1}{2}}} e^{\frac{4\pi}{c} \sqrt{\ell m}} \left(\frac{2\pi}{c} \sqrt{\ell m} \right)^{k-1} \leq \frac{(cdk\ell)^{O(1)}}{\Gamma(k-1)} e^{4\pi k \sqrt{\frac{\ell^{\frac{1}{2}}}{cd}}} \left(2\pi k \sqrt{\frac{\ell^{\frac{1}{2}}}{cd}} \right)^{k-1},$$

by bounding each term in the sum by the largest term. In the part of the sum with $m > \frac{k^2 c}{\ell^{\frac{1}{2}} d}$ we bound the Bessel function by $O(1)$ and W, \tilde{W} by $\ll \left(\frac{k^2}{80d^2 m} \right)^{k^{\frac{1}{4}}}$, yielding

$$\ll cd^{4\sigma} \sum_{m > k^2 c / \ell^{\frac{1}{2}} d} \frac{1}{m^{\frac{1}{2}+\sigma}} \left(\frac{k^2}{80md^2} \right)^{k^{1/4}} \ll cd^{4\sigma} \left(\frac{\ell^{\frac{1}{2}}}{80cd} \right)^{k^{1/4}}.$$

Summed over $cd > 10000\sqrt{\ell}$ both of these bounds converge and yield a result

$$\ll \ell^{O(1)} e^{-k^{1/4}},$$

which is negligible.

In order to apply the Voronoi summation formula to the sum over m we open the Kloosterman sum and introduce a function of compact support. Let $F \in C_c^\infty(\mathbb{R}^+)$ satisfy

- (1) $F(x) \equiv 1$ for $\frac{k}{1000} < x < 1000k\sqrt{\ell}$
- (2) $\text{supp}(F) \subset [\frac{k}{2000}, 2000k\sqrt{\ell}]$
- (3) For each $j = 0, 1, 2, \dots$ and all x , $x^j \frac{d^j}{dx^j} F(x) \ll_j 1$.

²The pole of ζ does not contribute since $\hat{H}(-\sigma) + \hat{H}(\sigma) = 0$.

and consider the perturbed sum

$$2\pi i^k \sum_{cd < 10000\sqrt{\ell}} \frac{1}{cd^{1+2\sigma}} \sum_{a \bmod c}^* e\left(\frac{\bar{a}\ell}{c}\right) \sum_{m=1}^{\infty} \frac{\tau_{it}(m)e\left(\frac{am}{c}\right)}{m^{\frac{1}{2}+\sigma}} \left[W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(md^2) \right] \\ J_{k-1}\left(\frac{4\pi}{c}\sqrt{\ell m}\right) F\left(\frac{4\pi}{c}\sqrt{\ell m}\right).$$

This negligibly changes the sum, since for those c, m for which $F\left(\frac{4\pi}{c}\sqrt{\ell m}\right)$ is not identically 1, either J_k or W or \tilde{W} is extremely small: there are $O(\ell^{O(1)}k^{O(1)})$ terms with $\frac{4\pi}{c}\sqrt{\ell m} < \frac{k}{1000}$ and for these terms, $J_{k-1}\left(\frac{4\pi}{c}\sqrt{\ell m}\right) \leq e^{-k}$. Meanwhile, if $\frac{4\pi}{c}\sqrt{\ell m} > 1000k\ell$ then $m > \left(\frac{1000}{4\pi}\right)^2 k^2 c^2$ so that the sum is bounded by

$$\ll \ell^{O(1)}k^{O(1)} \sum_{cd < 1000\sqrt{\ell}} \sum_{m > \left(\frac{1000}{4\pi}\right)^2 k^2 c^2} \left(\frac{k^2}{80md^2}\right)^{k^{\frac{1}{4}}} \ll \ell^{O(1)}k^{O(1)}e^{-k^{\frac{1}{4}}}.$$

Set

$$g(x) = g_{c,d}(x) = \frac{1}{x^{\frac{1}{2}+\sigma}} W_{k,\sigma+it}(d^2 x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\ell x}\right) F\left(\frac{4\pi}{c}\sqrt{\ell x}\right)$$

and

$$\tilde{g}_{c,d}(x) = \frac{1}{x^{\frac{1}{2}-\sigma}} \tilde{W}_{k,-\sigma+it}(d^2 x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\ell x}\right) F\left(\frac{4\pi}{c}\sqrt{\ell x}\right)$$

so that the summation over m is given by

$$\sum_m \tau_{it}(m) e\left(\frac{am}{c}\right) \{g_{c,d}(m) + (4\pi^2 d^2)^{2\sigma} \tilde{g}_{c,d}(m)\}.$$

Applying, for each c, d , Voronoi summation in the sum over m , the function $g_{c,d}$ yields four terms³

$$\begin{aligned} & 2\pi i^k \zeta(1-2it) \sum_{cd < 10000\ell^{\frac{1}{2}}} \frac{S(0, \ell; c)}{c^{2-2it} d^{1+2\sigma}} \int_0^\infty g_{c,d}(x) x^{-it} dx \\ & + 2\pi i^k \zeta(1+2it) \sum_{cd < 10000\ell^{\frac{1}{2}}} \frac{S(0, \ell; c)}{c^{2+2it} d^{1+2\sigma}} \int_0^\infty g_{c,d}(x) x^{it} dx \\ \text{(J)} \quad & + 2\pi i^k \sum_{cd < 10000\ell^{\frac{1}{2}}} \frac{1}{c^2 d^{1+2\sigma}} \sum_{n=1}^\infty \tau_{it}(n) S(0, \ell - n; c) \int_0^\infty g_{c,d}(x) J_{2it}^+\left(\frac{4\pi}{c}\sqrt{nx}\right) dx \\ \text{(K)} \quad & + 2\pi i^k \sum_{cd < 10000\ell^{\frac{1}{2}}} \frac{1}{c^2 d^{1+2\sigma}} \sum_{n=1}^\infty \tau_{it}(n) S(0, \ell + n; c) \int_0^\infty g_{c,d}(x) K_{2it}^+\left(\frac{4\pi}{c}\sqrt{nx}\right) dx. \end{aligned}$$

with four additional terms coming from summation against \tilde{g} . We are going to show that the first two terms combine with the corresponding terms from \tilde{g} to yield the remaining two main terms of the theorem, and that (J) and (K) are error terms.

³Note that summation over $a \bmod c^*$ has been replaced by Ramanujan sums.

4.1. The off-diagonal main terms. Expanding the definition of $g_{c,d}(x)$, the first two terms above are equal to

$$2i^k \Re \left\{ 2\pi \zeta(1-2it) \sum_{cd < 10000\ell^{\frac{1}{2}}} \frac{S(0, \ell; c)}{c^{2-2it} d^{1+2\sigma}} \times \int_0^\infty W_{k, \sigma+it}(d^2 x) J_{k-1} \left(\frac{4\pi}{c} \sqrt{\ell x} \right) F \left(\frac{4\pi}{c} \sqrt{\ell x} \right) x^{-\frac{1}{2}-\sigma-it} dx. \right\}$$

With negligible error the function F may be removed from the integrand, and then the sums extended to all c and d , this justified by the continuous analog of the arguments given above involving summations over m .⁴ Inserting the definition of $W_{k, \sigma+it}$ we obtain

$$2i^k \Re \left\{ 2\pi \zeta(1-2it) \sum_{c,d} \frac{S(0, \ell; c)}{c^{2-2it} d^{1+2\sigma}} \times \int_0^\infty \left[\frac{1}{2\pi i} \int_{(3)} \frac{1}{(4\pi^2 x d^2)^s} \frac{\Gamma(s + \sigma + \frac{k}{2} + it) \Gamma(s + \sigma + \frac{k}{2} - it)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds \right] \times J_{k-1} \left(\frac{4\pi}{c} \sqrt{\ell x} \right) x^{\frac{1}{2}-\sigma-it} \frac{dx}{x} \right\}$$

Substitute $w = \frac{4\pi}{c} \sqrt{\ell x}$ and exchange the order of the integration to obtain

$$2i^k \Re \left\{ \frac{(4\pi)^{2\sigma+2it} \zeta(1-2it)}{\ell^{\frac{1}{2}-\sigma-it}} \sum_{c,d} \frac{S(0, \ell; c)}{(cd)^{1+2\sigma}} \times \frac{1}{2\pi i} \int_{(3)} \left(\frac{4\ell}{c^2 d^2} \right)^s \frac{\Gamma(s + \sigma + \frac{k}{2} + it) \Gamma(s + \sigma + \frac{k}{2} - it)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) \left[\int_0^\infty J_{k-1}(w) w^{1-\sigma-2it-2s} \frac{dw}{w} \right] ds \right\}$$

We replace the inner integral with the Mellin transform of J_{k-1} . Also, we shift the sum over c and d under the integral. Recall that the Ramanujan sum evaluates to

$$\begin{aligned} S(0, a; p) &= -1, \\ S(0, a; p^e) &= 0, \\ S(0, p; p) &= p-1, & (a, p) = 1, \quad e \geq 1 \\ S(0, ap, p^2) &= -p, \\ S(0, ap, p^{e+1}) &= 0 \end{aligned}$$

Thus the resulting Dirichlet series $\sum_{c,d} \frac{S(0, \ell; c)}{(cd)^{1+2\sigma+2s}}$ collapses to the finite product

$$\prod_{p|\ell} \left(1 - \frac{1}{p^{1+2\sigma+2s}} \right)^{-1} \left(1 + \frac{p-1}{p^{1+2\sigma+2s}} - \frac{p}{p^{2+4\sigma+4s}} \right) = \prod_{p|\ell} \left(1 + \frac{1}{p^{2\sigma+2s}} \right) = \ell^{-\sigma-s} \tau_{s+\sigma}(\ell).$$

⁴We bound only the real part of the error. Recall that W and \tilde{W} are real, so that the imaginary parts of c^{it} and x^{it} are $O(t \log \ell)$ and $O(t \log x)$.

Combining these steps we arrive at

$$2i^k \Re \left\{ \frac{\zeta(1-2it)(2\pi)^{2\sigma+2it}}{\ell^{\frac{1}{2}-it}} \frac{1}{2\pi i} \int_{(3)} \tau_{s+\sigma}(\ell) \frac{\Gamma(\sigma + \frac{k}{2} - it + s) \Gamma(-\sigma + \frac{k}{2} - it - s)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds. \right\}$$

Repeating these steps, one proves that the main terms coming from $\tilde{g}_{c,d}$ are (again to within permissible error)

$$2i^k \Re \left\{ \frac{\zeta(1-2it)(2\pi)^{2\sigma+2it}}{\ell^{\frac{1}{2}-it}} \frac{1}{2\pi i} \int_{(3)} \tau_{-s+\sigma}(\ell) \frac{\Gamma(\sigma + \frac{k}{2} - it - s) \Gamma(-\sigma + \frac{k}{2} - it + s)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds \right\}.$$

In this integral, we change s to $-s$ and recall that $\hat{H}(-s) = -\hat{H}(s)$, so that the combined contribution from the $g_{c,d}$ and $\tilde{g}_{c,d}$ main terms is equal to

$$\frac{1}{2\pi i} \left\{ \int_{(3)} - \int_{(-3)} \right\} \tau_{s+\sigma}(\ell) \frac{\Gamma(\sigma + \frac{k}{2} - it + s) \Gamma(-\sigma + \frac{k}{2} - it - s)}{\Gamma(\sigma + \frac{k}{2} + it) \Gamma(\sigma + \frac{k}{2} - it)} \hat{H}(s) ds$$

Thus the two terms together are just equal to the residue of the integrand at the pole at 0, that is,

$$2i^k \Re \left\{ \zeta(1-2it)(2\pi)^{2\sigma+2it} \frac{\tau_{\sigma}(\ell)}{\ell^{\frac{1}{2}-it}} \frac{\Gamma(-\sigma + \frac{k}{2} - it)}{\Gamma(\sigma + \frac{k}{2} + it)} \right\} = 2\Re \left\{ \zeta(1-2it) \left(\frac{k}{4\pi} \right)^{-2\sigma-2it} \frac{\tau_{\sigma}(\ell)}{\ell^{\frac{1}{2}-it}} \right\} + O((1+t^2)k^{-1}).$$

4.2. The terms containing Bessel integrals. The term (K) is extremely small, since the K -Bessel function is exponentially small for large variable and the function $g_{c,d}$ localizes the variable to be of size at least $k\sqrt{\frac{n}{\ell}}$. The term (J) requires some more care, and we get cancelation from the change in rate of oscillation of the J -Bessel function in its transition region.

The integral in the term (J) is equal to

$$\int_0^\infty W_{k,\sigma+it}(d^2x) J_{k-1} \left(\frac{4\pi}{c} \sqrt{\ell x} \right) F \left(\frac{4\pi}{c} \sqrt{\ell x} \right) J_{2it}^+ \left(\frac{4\pi}{c} \sqrt{nx} \right) \frac{dx}{x^{\frac{1}{2}+\sigma}}$$

Substituting $y = \frac{4\pi}{c} \sqrt{\ell x}$ we obtain

$$(J) = \frac{i^k (4\pi)^{1+2\sigma}}{\ell^{\frac{1}{2}-\sigma}} \sum_{cd < 10000\sqrt{\ell}} \frac{1}{(cd)^{1+2\sigma}} \sum_{n=1}^{\infty} \tau_{it}(n) S(0, \ell - n; c) \times \int_0^\infty W_{k,\sigma+it} \left(\frac{c^2 d^2 y^2}{(4\pi)^2 \ell} \right) J_{k-1}(y) J_{2it}^+ \left(y \sqrt{\frac{n}{\ell}} \right) F(y) y^{-2\sigma} dy.$$

Now replace J_{2it}^+ with its asymptotic expansion

$$J_{2it}^+ \left(y \sqrt{\frac{n}{\ell}} \right) = -\sqrt{\frac{2\pi}{y}} \sqrt{\frac{\ell}{n}} \sin \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right) \left[1 - \frac{P(2it)\ell}{128y^2n} \right] - \pi \cos \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right) \frac{-4t^2 - \frac{1}{4}}{2y} + O \left(\frac{(1+t^6)\ell^{\frac{3}{2}}}{y^3 n^{\frac{3}{2}}} \right).$$

Using the integral bound in Lemma 3.5, the error term contributes $O(\frac{\ell^{2+\sigma}(1+t^6)}{k^{\frac{5}{2}+2\sigma-\epsilon}})$. In the remaining terms we can integrate by parts several times to truncate the sum over n at $n < \ell k^\epsilon$, with negligible error. We show only how to bound the contribution from integrating against the main term

$$(19) \quad -\sqrt{\frac{2\pi}{y}} \sqrt{\frac{\ell}{n}} \sin\left(y\sqrt{\frac{n}{\ell}} - \frac{\pi}{4}\right);$$

the rest of the main term can be handled in exactly the same way, and it produces an error of smaller size.

We will prove the following bound.

$$(B) \quad \int_0^\infty W_{k,\sigma+it} \left(\frac{c^2 d^2 y^2}{(4\pi)^2 \ell} \right) J_{k-1}(y) \sin\left(y\sqrt{\frac{n}{\ell}} - \frac{\pi}{4}\right) F(y) y^{\frac{-1}{2}-2\sigma} dy \ll \ell^{\frac{1}{2}} k^{-\frac{1}{2}-2\sigma+\epsilon}.$$

Assuming this bound for the moment we find that the contribution to (J) from integration against (19) is

$$\ll \frac{\ell^{\frac{1}{4}+\sigma}}{k^{\frac{1}{2}+2\sigma-\epsilon}} \sum_{cd < 1000\sqrt{\ell}} \frac{1}{(cd)^{1+2\sigma}} \sum_{n \ll \ell k^\epsilon} \frac{|S(0, \ell - n; c)|}{n^{\frac{1}{4}-\epsilon}}$$

Here the $n = \ell$ term contributes $\ll \ell^{\frac{1}{2}} k^{-\frac{1}{2}-2\sigma+\epsilon}$ while the $n \neq \ell$ terms give

$$\ll \frac{\ell^{\frac{1}{4}+\sigma}}{k^{\frac{1}{2}+2\sigma-\epsilon}} \sum_{\substack{n \ll \ell k^\epsilon \\ n \neq \ell}} \frac{1}{n^{\frac{1}{4}-\epsilon}} \sum_{\substack{c_1 | n - \ell \\ c_1 c_2 d \leq 1000\sqrt{\ell}}} \frac{1}{c_1^{2\sigma} c_2^{1+2\sigma} d^{1+2\sigma}} \ll \ell^{1+\sigma} k^{-\frac{1}{2}-2\sigma+\epsilon}.$$

The term corresponding to (J) coming from \tilde{g} is handled in an analogous way, so it only remains to prove the bound (B).

To prove the bound (B) we split the integral into the ranges $y < k - k^{1/3}$, $k - k^{1/3} < y < k + k^{1/3}$, and $k + k^{1/3} < y < 2000k\sqrt{\ell}$. For $y < k - k^{1/3}$ we use the bound

$$J_{k-1}(y) \ll \frac{e^{kw - k \tanh^{-1} w}}{\sqrt{kw}} + O(k^{-1}), \quad w = \sqrt{\frac{k^2}{x^2} - 1}$$

which easily gives a bound of $\ll k^{-\frac{1}{2}-2\sigma}$ since for small w ,

$$kw - k \tanh^{-1} w \sim \frac{kw^3}{3} \asymp k^{\frac{3\Delta-1}{2}}, \quad y = k - k^\Delta, \quad \Delta \geq \frac{1}{3}.$$

For $k - k^{1/3} < y < k + k^{1/3}$ we bound simply $J_{k-1}(y) \ll k^{-\frac{1}{3}}$, so that this part also contributes $\ll k^{-\frac{1}{2}-2\sigma}$.

In the remaining part of the integral we have from (15) (set $k' = k - 1$),

$$J_{k'}(x) = \sqrt{\frac{2}{\pi k' w}} \cos\left(k' w - k' \tan^{-1} w - \frac{\pi}{4}\right) + O(k^{-1} + k^{-1} w^{-2}); \quad w = \sqrt{\frac{x^2}{k'^2} - 1}.$$

Integrating the error term produces

$$\ll \sqrt{\ell} k^{-\frac{1}{2}-2\sigma} + \int_{k+k^{1/3}}^{1000k\sqrt{\ell}} \frac{1}{y^{\frac{1}{2}+2\sigma}} \frac{k'}{(y+k')(y-k')} dy \ll k^{-\frac{1}{2}-2\sigma} (\sqrt{\ell} + \log \ell).$$

Now consider a diadic interval $[k + A, k + 2A]$ with $A > k^{1/3}$. On such an interval we have that w is fixed to within a constant. Moreover,

$$\sqrt{\frac{2}{\pi k' w}} \cos \left(k' w - k' \tan^{-1} w - \frac{\pi}{4} \right) \sin \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right)$$

may be written as a linear combination of exponentials of the form

$$\sqrt{\frac{2}{\pi k' w}} e^{i[\pm(k' w - k' \tan^{-1} w - \frac{\pi}{4}) \pm (y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4})]} = G(y) e^{iF(y)}.$$

By further subdividing $[k + A, k + 2A]$ into $O(1)$ subintervals we may assume that $\frac{F'(y)}{G(y)}$ is monotonic. Recalling (16),

$$\frac{d^2}{dy^2} (k' w - k' \tan^{-1} w) = \frac{k'}{y^2 w},$$

we obtain from Lemma 3.4 that for each $B \in [k + A, k + 2A]$,

$$\int_{k+A}^B \sqrt{\frac{2}{\pi k' w}} \cos \left(k' w - k' \tan^{-1} w - \frac{\pi}{4} \right) \sin \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right) dy \ll \frac{k + A}{k'}.$$

Thus summing diadically we conclude that for all $z \in [k + k^{1/3}, 2000k\sqrt{\ell}]$ we have

$$I_z = \int_{k+k^{1/3}}^z \sqrt{\frac{2}{\pi k' w}} \cos \left(k' w - k' \tan^{-1} w - \frac{\pi}{4} \right) \sin \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right) dy \ll \frac{z \log \ell}{k'}.$$

Set

$$\begin{aligned} & \int_{k+k^{1/3}}^{2000k\sqrt{\ell}} W_{k,\sigma+it} \left(\frac{c^2 d^2 y^2}{(4\pi)^2 \ell} \right) \sqrt{\frac{2}{\pi k' w}} \cos \left(k' w - k' \tan^{-1} w - \frac{\pi}{4} \right) \sin \left(y \sqrt{\frac{n}{\ell}} - \frac{\pi}{4} \right) F(y) \frac{dy}{y^{\frac{1}{2}+2\sigma}} \\ &= \int_{k+k^{1/3}}^{2000k\sqrt{\ell}} W_{k,\sigma+it} \left(\frac{c^2 d^2 y^2}{(4\pi)^2 \ell} \right) F(y) y^{\frac{-1}{2}-2\sigma} dI_y \end{aligned}$$

and integrate by parts. Substituting our absolute bound for I_y and the bounds

$$\frac{\partial}{\partial y} W_{k,\sigma+it} \left(\frac{c^2 d^2 y^2}{(4\pi)^2 \ell} \right) \ll \frac{1}{y}, \quad F'(y) \ll \frac{1}{y}$$

gives the result $\ll k^{-\frac{1}{2}-2\sigma} \log \ell$. This completes the bound (B). \square

5. MOLLIFICATION

Write the inverse of the L function $L(s; f)$ as

$$L(s; f)^{-1} = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right), \quad \Re(s) > 1$$

The coefficients $a_f(n)$ are supported on cube-free numbers, and for m, n square-free, $(m, n) = 1$ we have $a_f(mn^2) = \mu(m) \lambda_f(m)$. We define a mollifier for $L(\frac{1}{2} + \sigma + it; f)$ by

$$(20) \quad M\left(\frac{1}{2} + \sigma + it; f\right) = \sum_{n=1}^{\infty} \frac{a_f(n) F(s(n))}{n^{\frac{1}{2}+\sigma+it}}.$$

Here $s(n) = \prod_{p|n} p$ denotes the squarefree kernel of n and $F(n)$ is a cut-off function to be given explicitly later, but for which we stipulate $F(n) \ll n^\epsilon$ and $F(n) = 0$ for $n > M = k^\theta$ for some $\theta < \frac{1}{6}$. In particular, we have the representation

$$M\left(\frac{1}{2} + \sigma + it; f\right) = \sum_{(m,n)=1} {}^b \frac{\mu(m)\lambda_f(m)F(mn)}{m^{\frac{1}{2}+\sigma+it}n^{1+2\sigma+2it}}$$

so that

$$\begin{aligned} \left| M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 &= \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1}} {}^b \frac{\mu(m_1)\mu(m_2)\lambda_f(m_1)\lambda_f(m_2)F(m_1n_1)F(m_2n_2)}{m_1^{\frac{1}{2}+\sigma+it}m_2^{\frac{1}{2}+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \\ (21) \quad &= \sum_d {}^b \frac{1}{d^{1+2\sigma}} \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1n_1m_2n_2, d)=1}} {}^b \frac{\mu(m_1)\mu(m_2)\lambda_f(m_1m_2)F(dm_1n_1)F(dm_2n_2)}{m_1^{\frac{1}{2}+\sigma+it}m_2^{\frac{1}{2}+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \end{aligned}$$

From this representation, we find

$$\begin{aligned} (22) \quad \sum_{f \in H_k}^h \left| M\left(\frac{1}{2} + \sigma + it; f\right) L\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 \\ = \sum_d {}^b \frac{1}{d^{1+2\sigma}} \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1n_1m_2n_2, d)=1}} {}^b \frac{\mu(m_1)\mu(m_2)F(m_1n_1d)F(m_2n_2d)}{m_1^{\frac{1}{2}+\sigma+it}m_2^{\frac{1}{2}+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \\ \times \sum_{f \in H_k}^h \lambda_f(m_1m_2) |L\left(\frac{1}{2} + \sigma + it; f\right)|^2 \end{aligned}$$

Substituting our expression for the twisted second moment, we find that

$$\begin{aligned} \text{expr. (22)} + O(k^{3\theta-2\sigma-\frac{1}{2}+\epsilon}) \\ = \sum_d {}^b \frac{1}{d^{1+2\sigma}} \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1n_1m_2n_2, d)=1}} {}^b \frac{\mu(m_1)\mu(m_2)F(m_1n_1d)F(m_2n_2d)}{m_1^{\frac{1}{2}+\sigma+it}m_2^{\frac{1}{2}+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \\ \times \left\{ \zeta(1+2\sigma) \frac{\tau_{it}(m_1m_2)}{(m_1m_2)^{\frac{1}{2}+\sigma}} + \zeta(1-2\sigma) \left(\frac{k}{4\pi}\right)^{-4\sigma} \frac{\tau_{it}(m_1m_2)}{(m_1m_2)^{\frac{1}{2}-\sigma}} \right. \\ \left. + i^k 2\Re \left[\zeta(1+2it) \left(\frac{k}{4\pi}\right)^{-2\sigma+2it} \frac{\tau_\sigma(m_1m_2)}{(m_1m_2)^{\frac{1}{2}+it}} \right] \right\} \\ = \mathcal{S}_1 + \mathcal{S}_2 + 2i^k \Re \mathcal{S}_3 \end{aligned}$$

We may rewrite the divisor sums

$$\tau_s(m_1m_2) = \sum_{\ell_1\ell_2=m_1m_2} \left(\frac{\ell_1}{\ell_2}\right)^s = \sum_{g|(m_1, m_2)} \mu(g) \tau_s\left(\frac{m_1}{g}\right) \tau_s\left(\frac{m_2}{g}\right).$$

Doing this, and shifting the sum over g to the front we separate the variables m_1 and m_2 . Thus we find:

$$\mathcal{S}_1 = \zeta(1+2\sigma) \sum_d \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+4\sigma}} \left| \sum_{\substack{(m,n)=1 \\ (mn,gd)=1}} \frac{\mu(m)\tau_{it}(m)F(mngd)}{m^{1+2\sigma+it}n^{1+2\sigma+2it}} \right|^2$$

and similar expressions for \mathcal{S}_2 , and \mathcal{S}_3 , although the inner sum in \mathcal{S}_3 is not a square. In fact, there is substantial cancellation in the inner summation for \mathcal{S}_1 above coming from the Möbius function. The sum is in fact equal to

$$\mathcal{S}_1 = \zeta(1+2\sigma) \sum_d \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+4\sigma}} \left| \sum_{(m,gd)=1} \frac{\mu(m)F(mgd)}{m^{1+2\sigma}} \right|^2$$

We also find

$$\begin{aligned} \mathcal{S}_2 &= \zeta(1-2\sigma) \left(\frac{k}{4\pi} \right)^{-4\sigma} \sum_d \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^2} \times \left| \sum_{\substack{(m,n)=1 \\ (mn,gd)=1}} \frac{\mu(m)\tau_{it}(m)F(mngd)}{m^{1+it}n^{1+2\sigma+2it}} \right|^2, \\ \mathcal{S}_3 &= \zeta(1+2it) \left(\frac{k}{4\pi} \right)^{-2\sigma+2it} \sum_d \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+2\sigma+2it}} \\ &\quad \times \sum_{(m_1m_2,gd)=1} \frac{\mu(m_1)F(m_1gd)F(m_2gd)}{m_1^{1+2it}} \sum_{m_2^1m_2^2m_2^3=m_2} \frac{\mu(m_2^1)\mu(m_2^2)}{(m_2^1)(m_2^2)^{1+2\sigma}(m_2^3)^{1+2\sigma-2it}}. \end{aligned}$$

5.1. Upper bound for the harmonic mollified second moment. We now fix the cut-off function F and prove an upper bound for the mollified second moment. Let

$$(23) \quad F(x) = \begin{cases} 1 & 0 \leq x \leq \sqrt{M} \\ P\left(\frac{\log(\frac{M}{x})}{\log M}\right) & \sqrt{M} \leq x \leq M \\ 0 & x \geq M \end{cases}$$

where $P(t) = 12t^2 - 16t^3$ satisfies $P(\frac{1}{2}) = 1$ and $P'(\frac{1}{2}) = P(0) = P'(0) = 0$. The function F is continuously differentiable. It's Mellin transform is equal to

$$(24) \quad \hat{F}(s) = \frac{24(M^s + M^{\frac{s}{2}})}{s^3(\log M)^2} - \frac{96(M^s - M^{\frac{s}{2}})}{s^4(\log M)^3}.$$

It has a simple pole at $s = 0$ with residue 1. Also, expanding $\hat{F}(s)$ in it's Laurent series about 0,

$$(25) \quad \hat{F}(s) = \frac{1}{s} + \sum_{n=0}^{\infty} c_n s^n$$

the coefficients c_n satisfy the bound

$$c_n \ll \frac{(\log M)^{n+1}}{(n+3)!}.$$

For this choice of cut-off function we prove

Proposition 5.1. *Let $M = k^\theta$ with $\theta < \frac{1}{6}$ and suppose $\frac{1}{\log k} < \sigma$ and $|t| < k^{\frac{1}{4}}$. For $M(\frac{1}{2} + \sigma + it; f)$ defined by (20) and cut-off function F as in (23) we have*

$$\sum_{f \in H_k}^h \left| M(\frac{1}{2} + \sigma + it; f) L(\frac{1}{2} + \sigma + it; f) \right|^2 \leq 1 + O(k^{3\theta - 2\sigma - \frac{1}{2} + \epsilon}) + O(k^{-\theta\sigma}).$$

Proof. We prove $\mathcal{S}_1 = 1 + O(K^{-\theta\sigma})$ and $\Re(\mathcal{S}_3) = O(K^{-\theta\sigma})$. This suffices because $\mathcal{S}_2 \leq 0$ since $\zeta(1 - 2\sigma) < 0$.

By Mellin inversion

$$(26) \quad \mathcal{S}_1 = \zeta(1 + 2\sigma) \left(\frac{1}{2\pi i} \right)^2 \int_{(2)} \int_{(2)} \hat{F}(\alpha) \hat{F}(\beta) G(\alpha, \beta; \sigma) d\alpha d\beta$$

where

$$\begin{aligned} G(\alpha, \beta; \sigma) &= \sum_d^b \frac{1}{d^{1+2\sigma+\alpha+\beta}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+4\sigma+\alpha+\beta}} \sum_{\substack{(m_1, gd)=1 \\ (m_2, gd)=1}}^b \frac{\mu(m_1)\mu(m_2)}{m_1^{1+2\sigma+\alpha} m_2^{1+2\sigma+\beta}} \\ &= \prod_p (1 - p^{-1-2\sigma-\alpha} - p^{-1-2\sigma-\beta} + p^{-1-2\sigma-\alpha-\beta}) \\ &= \frac{\zeta(1 + 2\sigma + \alpha + \beta)}{\zeta(1 + 2\sigma + \alpha)\zeta(1 + 2\sigma + \beta)} H(\alpha, \beta; \sigma) \end{aligned}$$

The Euler product defining H converges absolutely in the region $\alpha + 2\sigma > -\frac{1}{2}$, $\beta + 2\sigma > -\frac{1}{2}$, $\alpha + \beta + 2\sigma > -\frac{1}{2}$.

To evaluate the integral, shift both contours to the line $\Re(\alpha) = \Re(\beta) = \frac{1}{\log k}$ and truncate the β integral at $|\Im(\beta)| \leq k$ with error $O(k^{-2+\epsilon})$. Then shift the α integral to the contour \mathcal{C} given by

$$\mathcal{C} := \{\alpha : \Re(\alpha) = -2\sigma - \log^{3/4}(2 + |\Im(\alpha)|)\}.$$

In shifting the α contour to \mathcal{C} we encounter poles at $\alpha = 0$ and $\alpha = -2\sigma - \beta$. This first pole yields a residue

$$(27) \quad \frac{1}{\zeta(1 + 2\sigma)} \frac{1}{2\pi i} \int_{\frac{1}{\log k} - ik}^{\frac{1}{\log k} + ik} \hat{F}(\beta) d\beta = \frac{1 + O(k^{-2})}{\zeta(1 + 2\sigma)}.$$

The second pole has residue

$$\frac{1}{2\pi i} \int_{1 + \frac{1}{\log k} - ik}^{1 + \frac{1}{\log k} + ik} \hat{F}(\beta) \hat{F}(-2\sigma - \beta) \frac{H(-2\sigma - \beta, \beta; \sigma)}{\zeta(1 - \beta)\zeta(1 + 2\sigma + \beta)} d\beta$$

Here we can extend the integration to the full line, and shift the contour to $\Re(\beta) = -\sigma$. On this line, $H(-\sigma + is, -\sigma - is; \sigma)$ is uniformly bounded, and so the integral is bounded by

$$(28) \quad \int_{-\infty}^{\infty} \left| \frac{\hat{F}(-\sigma + is)}{\zeta(1 + \sigma - is)} \right|^2 ds \\ \ll M^{-\sigma} \left[\int_{-1}^1 \left| \frac{1}{|\sigma + is|^2 (\log M)^2} + \frac{1}{|\sigma + is|^3 (\log M)^3} \right|^2 ds + O\left(\frac{1}{(\log M)^4}\right) \right].$$

Now using $(a + b)^2 \leq 2(a^2 + b^2)$, the right hand side is bounded by

$$k^{-\theta\sigma} \left[O\left(\frac{1}{(\log k)^4}\right) + \frac{1}{(\log M)^4} \int_{-\infty}^{\infty} \frac{ds}{(s^2 + \sigma^2)^2} + \frac{1}{(\log M)^6} \int_{-\infty}^{\infty} \frac{ds}{(s^2 + \sigma^2)^3} \right]$$

Since $\sigma \geq \frac{1}{\log k}$ we deduce that the second residue is $\ll \frac{k^{-\theta\sigma}}{\log k}$. The remaining integral, for α on \mathcal{C} , is bounded using standard bounds for ζ in the zero-free region and is quite small. Since with $\zeta(1 + 2\sigma) \ll \log k$ we have the claimed evaluation of \mathcal{S}_1 .

In bounding $2\Re(\mathcal{S}_3)$ we handle separately the cases $t \leq \frac{1}{4\log K}$ and $t > \frac{1}{4\log K}$.

When $t > \frac{1}{4\log K}$ we bound \mathcal{S}_3 in magnitude as we did \mathcal{S}_1 . By Mellin inversion

$$(29) \quad \mathcal{S}_3 = \zeta(1 + 2it) \left(\frac{k}{4\pi}\right)^{-2\sigma+2it} \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \int_{(2)} \hat{F}(\alpha) \hat{F}(\beta) G(\alpha, \beta; \sigma, t) d\alpha d\beta$$

where now $G(\alpha, \beta; \sigma, t)$ is given by

$$G(\alpha, \beta; \sigma, t) = \sum_d \frac{1}{d^{1+2\sigma+\alpha+\beta}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+2\sigma+2it+\alpha+\beta}} \\ \times \sum_{(m_1 m_2, gd)=1} \frac{\mu(m_1)}{m_1^{1+2it+\alpha}} \sum_{m_2^1 m_2^2 m_2^3 = m_2} \frac{\mu(m_2^1) \mu(m_2^2)}{(m_2^1)^{1+\beta} (m_2^2)^{1+2\sigma+\beta} (m_2^3)^{1+2\sigma-2it+\beta}} \\ = \prod_p \left[1 - \frac{1}{p^{1+\beta}} - \frac{1}{p^{1+2\sigma+\beta}} + \frac{1}{p^{1+2\sigma-2it+\beta}} - \frac{1}{p^{1+2it+\alpha}} + \frac{1}{p^{1+2\sigma+\alpha+\beta}} \right. \\ \left. + \frac{1}{p^{2+2it+\alpha+\beta}} - \frac{1}{p^{2+2\sigma+\alpha+\beta}} \right] \\ = \frac{\zeta(1 + 2\sigma - 2it + \beta) \zeta(1 + 2\sigma + \alpha + \beta)}{\zeta(1 + \beta) \zeta(1 + 2\sigma + \beta) \zeta(1 + 2it + \alpha)} H(\alpha, \beta; \sigma, t)$$

Here the Euler product defining $H(\alpha, \beta; \sigma, t)$ converges absolutely for

$$\min(\Re(\alpha), \Re(\beta), \Re(\alpha + \beta)) > -\frac{1}{2}.$$

To evaluate the integral in (29), shift α and β contours to the lines $\Re(\alpha) = \Re(\beta) = \frac{1}{\log k}$. We may assume that $\sigma < \frac{100 \log \log k}{\log k}$ since otherwise $k^{-2\sigma} < \frac{k^{-\sigma}}{(\log k)^{100}}$ and the integral may

be bounded directly using standard bounds for ζ and ζ^{-1} to the right of the 1-line. Now truncate the β contour at $|\Im(\beta)| < k$ and shift the α contour to \mathcal{C}' given by

$$\mathcal{C}' := \{\alpha : \Re(\alpha) = -\log^{3/4}(2 + |\Im(\alpha + 2it)|)\}.$$

In doing so we pass two poles, at $\alpha = 0$ and at $\alpha = -\beta - 2\sigma$. The first pole has residue

$$\begin{aligned} & \zeta(1 + 2it)^{-1} \frac{1}{2\pi i} \int_{\frac{1}{\log k} - ik}^{\frac{1}{\log k} + ik} \frac{\hat{F}(\beta) \zeta(1 + 2\sigma - 2it + \beta)}{\zeta(1 + \beta)} H(0, \beta; \sigma, t) d\beta \\ &= \zeta(1 + 2it)^{-1} \left(\frac{\hat{F}(-2\sigma + 2it)}{\zeta(1 - 2\sigma + 2it)} H(0, -2\sigma + 2it; \sigma, t) + O(1) \right). \end{aligned}$$

Expressing $\hat{F}(-2\sigma + 2it)$ using either the Laurent expansion for (25) for $|t| < \frac{1}{\log k}$ or the direct definition (23) for $|t| > \frac{1}{\log k}$, together with the bound $\frac{1}{\zeta(1-s)} \ll s$ valid in the standard zero-free region we have that this residue is $O(\zeta(1 + 2it)^{-1})$.

The second residue is equal to

$$\frac{1}{2\pi i} \int_{\frac{1}{\log k} - ik}^{\frac{1}{\log k} + ik} \hat{F}(-\beta - 2\sigma) \hat{F}(\beta) \frac{\zeta(1 + 2\sigma - 2it + \beta)}{\zeta(1 + \beta) \zeta(1 + 2\sigma + \beta) \zeta(1 + 2it - 2\sigma - \beta)} H(-2\sigma - \beta, \beta) d\beta$$

Shifting this integral to the line $\Re(\beta) = -2\sigma$ (the horizontal integrals are very small), and taking absolute values, we obtain a bound

$$\ll \int_{-k}^k \frac{|\hat{F}(-2\sigma + is)|}{|\zeta(1 - 2\sigma + is)|} \frac{|\hat{F}(-is)|}{|\zeta(1 - is)|} ds.$$

Arguing as above we have for all real s , $\frac{|\hat{F}(-is)|}{|\zeta(1 - is)|} = O(1)$ while for $s \in [-k, k]$,

$$\frac{|\hat{F}(-2\sigma + is)|}{|\zeta(1 - 2\sigma + is)|} \ll M^{-\sigma} \left[\frac{1}{(\log M)^2 |\sigma + is|^2} + \frac{1}{(\log M)^3 |\sigma + is|^3} \right].$$

so that the integral is $O(\frac{k^{-\theta\sigma}}{\log k})$ as in the second residue calculation for \mathcal{S}_1 . The remaining double integral with α on the contour $\mathcal{C} - 2it$ is again small. Thus for $\frac{1}{4\log k} < t$, we have

$$O(\zeta(1 + it)^{-1}) + O\left(\frac{k^{-\theta\sigma}}{\log k}\right)$$

for the integral in (29), which suffices since in this range, $\zeta(1 + 2it) = O(\log k)$.

When $|t| < \frac{1}{4\log k}$, we bound $2\Re\mathcal{S}_3$ to balance the fact that $\zeta(1 + 2it)$ can be quite large (but mostly imaginary). Following the method of [2], let \mathcal{O} be the circle $|w| = \frac{1}{2\log k}$. By Cauchy's residue theorem

$$2\Re\mathcal{S}_3 = \frac{1}{2\pi i} \int_{\mathcal{O}} \left(\frac{k}{4\pi}\right)^{-2\sigma+w} \zeta(1 + w) \eta(w; \sigma) \left[\frac{1}{w + 2it} + \frac{1}{w - 2it} \right] dw$$

with

$$\begin{aligned} \eta(w; \sigma) &= \sum_d \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+2\sigma+w}} \\ &\quad \times \sum_{(m_1 m_2, gd)=1} \frac{\mu(m_1) F(m_1 gd) F(m_2 gd)}{m_1^{1+w}} \sum_{m_1^1 m_2^2 m_3^3 = m_2} \frac{\mu(m_2^1) \mu(m_2^2)}{(m_2^1)(m_2^2)^{1+2\sigma} (m_2^3)^{1+2\sigma-w}} \end{aligned}$$

As before, we may assume that $\sigma < \frac{100 \log \log k}{\log k}$. The evaluation of $\eta(w; \sigma)$ by Mellin inversion is exactly analogous to the integral performed in calculating \mathcal{S}_3 when $\frac{1}{4 \log k} < |t|$: there is a main term equal to $\zeta(1+w)^{-1} O(1)$, a secondary residue term of size $\frac{M^{-\sigma}}{\log k}$ and a smaller error integral. Thus $\eta(w; \sigma) = O(\frac{1}{\log k})$. Thus the integrand in the integral over \mathcal{O} is $O(k^{-\theta\sigma} \log k)$. Since the length of the integral is $O(\frac{1}{\log k})$ the integral itself is $O(k^{-\theta\sigma})$. \square

6. REMOVING THE HARMONIC WEIGHTS

The starting point for the Kowalski-Michel [13] method for removing harmonic weights is the formula ([10])

$$w_f^{-1} = \frac{L(1, \text{sym}^2 f)}{\zeta(2)} |H_k| + O(\log^3 k)$$

where $L(s, \text{sym}^2 f)$ is the symmetric square L -function associated to f , defined by

$$L(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\rho_f(n)}{n^s} = \zeta(2s) \sum_n \frac{\lambda_f(n^2)}{n^s}.$$

Thus the natural average is expressed as

$$\begin{aligned} \frac{1}{|H_k|} \sum_{f \in H_k} \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 &= \frac{1}{|H_k|} \sum_{f \in H_k}^h w_f^{-1} \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 \\ &= \frac{1}{\zeta(2)} \sum_{f \in H_k}^h L(1, \text{sym}^2 f) \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 + O(k^{-1+\epsilon}). \end{aligned}$$

The method replaces $L(1, \text{sym}^2 f)$ with a short Dirichlet polynomial approximation

$$w_f(x) = \sum_{n \leq x} \frac{\rho_f(n)}{n}, \quad x = k^\kappa.$$

A minor modification to the proof of Proposition 2 of [13] yields the following result.

Proposition 6.1. *Assume that the mollifier $M(\frac{1}{2} + \sigma + it; f)$ is such that*

$$(30) \quad \sup_{f \in H_k} w_f \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 < k^{-\delta}, \quad \delta > 0$$

and

$$(31) \quad \sum_{f \in H_k}^h \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 < (\log k)^A.$$

Let $x = k^\kappa$ for some $\kappa > 0$. Then there is a $\gamma = \gamma(\delta, \kappa, A) > 0$ such that

$$\frac{1}{|H_k|} \sum_{f \in H_k} \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 = \sum_{f \in H_k}^h w_f(x) \left| L \cdot M\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 + O(k^{-\gamma}).$$

The result of the previous section guarantees condition (31). Trivially $|M(\frac{1}{2} + \sigma + it)| < k^{\frac{\theta}{2} + \epsilon}$ and the best known subconvex bound (see [11]) gives $L(\frac{1}{2} + \sigma + it) \ll (k + |t|)^{\frac{1}{3} - \frac{2\sigma}{3} + \epsilon}$. Thus condition (30) holds uniformly in $|t| < k$ for $\theta < \frac{1}{3}$. Therefore, we complete the proof of Proposition 2.2 by proving the following uniform bound.

Proposition 6.2. *For sufficiently small $\kappa, \delta, \theta > 0$ we have, uniformly in $\frac{1}{\log k} < \sigma \leq 1$ and $|t| < k^\delta$,*

$$\frac{1}{\zeta(2)} \sum_{f \in H_k}^h \left(\sum_{n \leq x = k^\kappa} \frac{\rho_f(n)}{n} \right) \left| M \cdot L\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 \leq 1 + O(k^{-\theta\sigma} + k^{-\kappa/2 + \epsilon}),$$

where M is the mollifier from the previous section, having length $M = k^\theta$.

6.1. Proof of Proposition 6.2. Combining expression (21) for $|M(\frac{1}{2} + \sigma + it; f)|^2$ with $\sum_{n \leq x} \frac{\rho_f(n)}{n} = \sum_{\ell^2 d < x} \frac{\lambda_f(d^2)}{\ell^2 d}$ and the Hecke relations, we obtain

$$\begin{aligned} & \sum_{n \leq x} \frac{\rho_f(n)}{n} |M(\frac{1}{2} + \sigma + it; f)|^2 \\ &= \sum_g^b \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1 m_2 n_1 n_2, g)=1}} \frac{\mu(m_1) \mu(m_2) F(m_1 n_1 g) F(m_2 n_2 g)}{m_1^{\frac{1}{2} + \sigma + it} m_2^{\frac{1}{2} + \sigma - it} n_1^{1 + 2\sigma + 2it} n_2^{1 + 2\sigma - 2it}} \sum_{\ell^2 d < x} \frac{1}{\ell^2 d} \sum_{h|(d^2, m_1 m_2)} \lambda_f\left(\frac{m_1 m_2 d^2}{h^2}\right). \end{aligned}$$

Write $h = h_1 h_2^2$ where h_1 and h_2 are squarefree. Clearly $h_2 | (m_1, m_2)$ and $h_2 | d$. Also, $h_1 | (d, m_1 m_2)$. Shifting the orders of summation, and then introducing our expression for the twisted second moment, we obtain

$$\begin{aligned} & \sum_{f \in H_k}^h \left(\sum_{n \leq x = k^\kappa} \frac{\rho_f(n)}{n} \right) \left| M \cdot L\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 \\ &= \sum_\ell \frac{1}{\ell^2} \sum_g^b \frac{1}{g^{1+2\sigma}} \sum_{(h_2, g)=1}^b \frac{1}{h_2^{2+2\sigma}} \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1 m_2 n_1 n_2, g h_2)=1}} \frac{\mu(m_1) \mu(m_2) F(m_1 n_1 g h_2) F(m_2 n_2 g h_2)}{m_1^{\frac{1}{2} + \sigma + it} m_2^{\frac{1}{2} + \sigma - it} n_1^{1 + 2\sigma + 2it} n_2^{1 + 2\sigma - 2it}} \sum_{h_1 | m_1 m_2}^b \frac{1}{h_1} \\ & \quad \times \sum_{d < \frac{x}{\ell^2 h_1 h_2}} \left\{ \zeta(1 + 2\sigma) \frac{\tau_{it}(m_1 m_2 d^2)}{d(m_1 m_2 d^2)^{\frac{1}{2} + \sigma}} + \zeta(1 - 2\sigma) \left(\frac{k}{4\pi}\right)^{-4\sigma} \frac{\tau_{it}(m_1 m_2 d^2)}{d(m_1 m_2 d^2)^{\frac{1}{2} - \sigma}} \right. \\ & \quad \left. + 2i^k \Re \left(\zeta(1 + 2it) \left(\frac{k}{4\pi}\right)^{-2\sigma + 2it} \frac{\tau_\sigma(m_1 m_2 d^2)}{d(m_1 m_2 d^2)^{\frac{1}{2} + it}} \right) + O\left(k^{-\frac{1}{2} - 2\sigma + \epsilon} d^{1+2\sigma} (m_1 m_2)^{1+\sigma}\right) \right\} \end{aligned}$$

The error term contributes

$$\ll k^{-1/2 + 3\theta - 2\sigma + \epsilon} x^{2+2\sigma} \ll k^{-1/2 + 3\theta + \epsilon} x^2.$$

For $\sigma > \frac{1}{4}$, the terms involving $\zeta(1 - 2\sigma)$ and $\zeta(1 + 2it)$ are negligibly small and so we are left to consider only the $\zeta(1 + 2\sigma)$ term; otherwise, for $\frac{1}{\log k} < \sigma < \frac{1}{4}$ we consider all three terms. In either case, we may remove the restriction on the sum over d with error $\ll x^{-\frac{1}{2}+\epsilon}$. Thus

$$\frac{1}{\zeta(2)} \sum_{f \in H_k}^h \left(\sum_{n \leq x=k^\kappa} \frac{\rho_f(n)}{n} \right) \left| M \cdot L\left(\frac{1}{2} + \sigma + it; f\right) \right|^2 = \mathcal{S}_1 + \mathcal{S}_2 + 2\Re \mathcal{S}_3$$

with the stipulation that $\mathcal{S}_2 = \mathcal{S}_3 = 0$ if $\sigma > \frac{1}{4}$.

We use the following lemma.

Lemma 6.3. *Let m_1 and m_2 be squarefree. For $\Re(s \pm \gamma) > 1$ we have*

$$\sum_d \frac{\tau_\gamma(m_1 m_2 d^2)}{d^s} = \frac{\zeta(s)}{\zeta(2s)} \zeta(s + 2\gamma) \zeta(s - 2\gamma) \prod_{p | \frac{m_1 m_2}{(m_1, m_2)^2}} \frac{p^\gamma + p^{-\gamma}}{1 + p^{-s}} \prod_{p | (m_1, m_2)} \frac{1 + p^{2\gamma} + p^{-2\gamma} - p^{-s}}{1 + p^{-s}}.$$

We first prove that for $\sigma < \frac{1}{4}$, $\mathcal{S}_2 < 0$, so that it may be discarded. Since we assume $\sigma < \frac{1}{4}$,

$$\begin{aligned} \mathcal{S}_2 &= \zeta(1 - 2\sigma) \frac{\zeta(2 - 2\sigma)}{\zeta(4 - 4\sigma)} |\zeta(2 - 2\sigma + 2it)|^2 \sum_{k=gh}^b \frac{1}{g^{1-2\sigma} h^{2-2\sigma}} \\ &\times \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1 n_1 m_2 n_2, k)=1}}^b \frac{\mu(m_1) \mu(m_2) F(m_1 n_1 k) F(m_2 n_2 k)}{m_1^{1+it} m_2^{1-it} n_1^{1+2\sigma+2it} n_2^{1+2\sigma-2it}} \prod_{p|m_1} \left(\left(1 + \frac{1}{p}\right) \left(\frac{p^{it} + p^{-it}}{1 + p^{-2+2\sigma}}\right) \right) \\ &\times \prod_{p|m_2} \left(\left(1 + \frac{1}{p}\right) \left(\frac{p^{it} + p^{-it}}{1 + p^{-2+2\sigma}}\right) \right) \prod_{p|(m_1, m_2)} \left(\frac{p}{p+1} \frac{(1 + p^{2it} + p^{-2it} - p^{-2+2\sigma})(1 + p^{-2+2\sigma})}{(p^{it} + p^{-it})^2} \right) \end{aligned}$$

This may be rearranged as

$$\begin{aligned} &\zeta(1 - 2\sigma) \frac{\zeta(2 - 2\sigma)}{\zeta(4 - 4\sigma)} |\zeta(2 - 2\sigma + 2it)|^2 \\ &\times \sum_{k=ghr}^b \frac{a(r)}{g^{1-2\sigma} h^{2-2\sigma} r^{2\sigma}} \left| \sum_{\substack{(m_1, n_1)=1 \\ (m_1 n_1, k)=1}}^b \frac{\mu(m_1) F(m_1 n_1 k)}{m_1^{1+it} n_1^{1+2\sigma+2it}} \prod_{p|m_1} \left(\left(1 + \frac{1}{p}\right) \left(\frac{p^{it} + p^{-it}}{1 + p^{-2+2\sigma}}\right) \right) \right|^2, \end{aligned}$$

where $a(r)$ is the multiplicative function, supported on squarefree integers, and given on primes by

$$\begin{aligned} a(p) &= \frac{p+1}{p} \frac{1 + p^{2it} + p^{-2it} - p^{-2+2\sigma}}{1 + p^{-2+2\sigma}} - \left(\frac{p+1}{p} \frac{p^{it} + p^{-it}}{1 + p^{-2+2\sigma}} \right)^2 \\ &= -\frac{p+1}{p} - (p^{it} + p^{-it})^2 \left[\left(\frac{p+1}{p + p^{-1+2\sigma}} \right)^2 - \frac{p+1}{p + p^{-1+2\sigma}} \right]. \end{aligned}$$

Now observe

$$\sum_{ghr=k} \frac{a(r)}{g^{1-2\sigma} h^{1-2\sigma} r^2} = \prod_{p|k} b(p); \quad b(p) = \frac{1}{p^{1-2\sigma}} + \frac{1}{p^{2-2\sigma}} + \frac{a(p)}{p^2}.$$

We have $b(p) \geq 0$; indeed, it suffices to check this under the conditions $|p^{it} + p^{-it}| = 2$, $\sigma = 0$ and $p = 2$, and in this case we find a value of 0.135. In particular, since $\zeta(1-2\sigma) < 0$ this proves that $\mathcal{S}_2 \leq 0$.

Next we turn to \mathcal{S}_1 . We have

$$\begin{aligned} \mathcal{S}_1 &= \zeta(1+2\sigma) \frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)} |\zeta(2+2\sigma+2it)|^2 \sum_{k=gh}^b \frac{1}{g^{1+2\sigma} h^{2+2\sigma}} \\ &\times \sum_{\substack{(m_1, n_1)=1 \\ (m_2, n_2)=1 \\ (m_1 n_1 m_2 n_2, k)=1}}^b \frac{\mu(m_1) \mu(m_2) F(m_1 n_1 k) F(m_2 n_2 k)}{m_1^{1+2\sigma+it} m_2^{1+2\sigma-it} n_1^{1+2\sigma+2it} n_2^{1+2\sigma-2it}} \prod_{p|m_1} \left(\left(1 + \frac{1}{p}\right) \left(\frac{p^{it} + p^{-it}}{1 + p^{-2-2\sigma}} \right) \right) \\ &\times \prod_{p|m_2} \left(\left(1 + \frac{1}{p}\right) \left(\frac{p^{it} + p^{-it}}{1 + p^{-2-2\sigma}} \right) \right) \prod_{p|(m_1, m_2)} \left(\frac{p}{p+1} \frac{(1 + p^{2it} + p^{-2it} - p^{-2-2\sigma})(1 + p^{-2-2\sigma})}{(p^{it} + p^{-it})^2} \right) \\ &= \zeta(1+2\sigma) \frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)} |\zeta(2+2\sigma+2it)|^2 \left(\frac{1}{2\pi i} \right)^2 \int_{(2)} \int_{(2)} \hat{F}(\alpha) \hat{F}(\beta) G(\alpha, \beta; \sigma, t) d\alpha d\beta; \\ G(\alpha, \beta; \sigma, t) &= \frac{\zeta(4+4\sigma)}{\zeta(2+2\sigma)} \prod_p \left[1 + \frac{1}{p^{2+2\sigma}} + \frac{1}{p^{1+2\sigma+\alpha+\beta}} + \frac{1}{p^{2+2\sigma+\alpha+\beta}} + \frac{1}{p^{4+4\sigma+\alpha+\beta}} - \frac{1}{p^{5+6\sigma+\alpha+\beta}} \right. \\ &\quad - \frac{1}{p^{1+2\sigma+\alpha}} - \frac{1}{p^{2+2\sigma+\alpha}} - \frac{1}{p^{2+2\sigma+2it+\alpha}} + \frac{1}{p^{3+4\sigma+2it+\alpha}} - \frac{1}{p^{1+2\sigma+\beta}} \\ &\quad \left. - \frac{1}{p^{2+\sigma+\beta}} - \frac{1}{p^{2+2\sigma-2it+\beta}} + \frac{1}{p^{3+4\sigma-2it+\beta}} \right] \end{aligned}$$

Here

$$G(\alpha, \beta; \sigma, t) = \frac{\zeta(1+2\sigma+\alpha+\beta)}{\zeta(1+2\sigma+\alpha)\zeta(1+2\sigma+\beta)} \tilde{H}(\alpha, \beta; \sigma, t)$$

where \tilde{H} is given by an absolutely convergent Euler product for

$$\min(\Re(\alpha), \Re(\beta), \Re(\alpha+\beta)) > -2\sigma - c$$

for some $c > 0$. This is to say that the contour giving \mathcal{S}_1 under the natural average is the same as for the harmonic average up to a change in the absolutely convergent Euler product. Thus the analysis of \mathcal{S}_1 from the previous section goes through without change to give

$$\mathcal{S}_1 = \zeta(1+2\sigma) \frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)} |\zeta(2+2\sigma+2it)|^2 G(0, 0; \sigma, t) + O(k^{-\theta\sigma}) = 1 + O(k^{-\theta\sigma}).$$

It is again the case that the contour integral giving \mathcal{S}_3 is the same for the natural average as for the harmonic average, up to an absolutely convergent Euler product. Thus the analysis of the previous section yields the bound $\Re(\mathcal{S}_3) = O(k^{-\theta\sigma})$, which completes the proof of Proposition 6.2.

REFERENCES

- [1] E. Bombieri and D. A. Hejhal. On the distribution of zeros of linear combinations of Euler products. *Duke Math. J.*, 80(3):821–862, 1995.
- [2] J. B. Conrey and K. Soundararajan. Real zeros of quadratic Dirichlet L -functions. *Invent. Math.*, 150(1):1–44, 2002.
- [3] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.
- [4] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. II*. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.
- [5] R. F. Faiziev. Estimates in the mean in the additive divisor problem. *Izv. Akad. Nauk Tadzhik. SSR Otdel. Fiz.-Mat. Khim. i Geol. Nauk*, (1(95)):7–17, 1985.
- [6] O. M. Fomenko. Automorphic L -functions in the weight aspect. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 314(Anal. Teor. Chisel i Teor. Funkts. 20):221–246, 289–290, 2004.
- [7] B. Hough. The distribution of central values of modular form L -functions. Preprint available.
- [8] A. E. Ingham. On the estimation of $N(\sigma, T)$. *Quart. J. Math., Oxford Ser.*, 11:291–292, 1940.
- [9] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
- [10] Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak. Low lying zeros of families of L -functions. *Inst. Hautes Études Sci. Publ. Math.*, (91):55–131 (2001), 2000.
- [11] Matti Jutila and Yoichi Motohashi. Uniform bound for Hecke L -functions. *Acta Math.*, 195:61–115, 2005.
- [12] Rizwanur Khan. The first moment of the symmetric-square L -function. *J. Number Theory*, 124(2):259–266, 2007.
- [13] E. Kowalski and P. Michel. The analytic rank of $J_0(q)$ and zeros of automorphic L -functions. *Duke Math. J.*, 100(3):503–542, 1999.
- [14] N. V. Kuznetsov. Convolution of Fourier coefficients of Eisenstein-Maass series. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 129:43–84, 1983. Automorphic functions and number theory. I.
- [15] Wen Zhi Luo. Zeros of Hecke L -functions associated with cusp forms. *Acta Arith.*, 71(2):139–158, 1995.
- [16] G. Ricotta. Real zeros and size of Rankin-Selberg L -functions in the level aspect. *Duke Math. J.*, 131(2):291–350, 2006.
- [17] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L -functions: symplectic and orthogonal examples. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 293–303. Amer. Math. Soc., Providence, RI, 2006.
- [18] Atle Selberg. Contributions to the theory of Dirichlet’s L -functions. *Skr. Norske Vid. Akad. Oslo. I.*, 1946(3):62, 1946.
- [19] Atle Selberg. Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.*, 48(5):89–155, 1946.
- [20] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

E-mail address: rdhough@math.stanford.edu